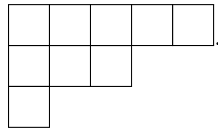
Plancherel Pie

π Day 2024

(Alex Wilson, 204 King)

For a fixed positive integer n, the *Plancherel measure* is a probability measure on the set of *partitions*[1] of n (weakly decreasing sequences of positive integers summing to n). We represent these partitions as rows of left-justified boxes—for example, the partition $(5, 3, 1)$ looks like



A filling of these boxes with the numbers $1, \ldots, n$ such that the rows increase left-to-right and columns increase top-to-bottom is called a *standard Young tableau*. For example,

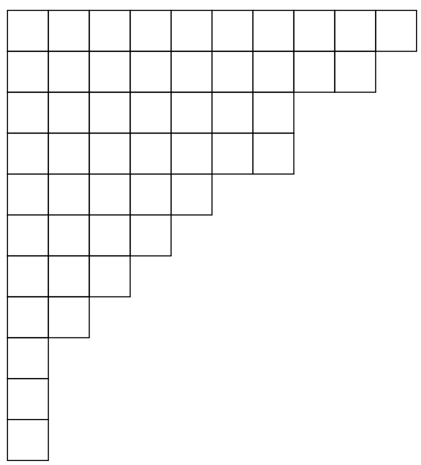
1	3	4	6	7
2	5	8		
9				

is a standard Young tableau of shape $(5, 3, 1)$. In the Plancherel measure, a particular partition λ appears with probability

$$\mu(\lambda) = \frac{(f^\lambda)^2}{n!}$$

where f^λ is the number of standard Young tableaux of shape λ. That is, the more possible fillings there are of a shape, the more likely it is to appear.

This pie is decorated with a random partition of 50. Out of the 204,226 possibilities I sampled $(10, 9, 7, 7, 5, 4, 3, 2, 1, 1, 1)$:



But how do we choose a partition at random, making sure each λ shows up with probability $\mu(\lambda)$?

Sampling in the Plancherel measure

The letter π isn't just a number—it also often represents a permutation! To get a random partition of 50 with the appropriate weighted probability, you should first sample a *permutation* of 50 elements uniformly at random. Then the RSK algorithm allows you to turn this permutation into a standard Young tableau. If you then take the shape of that tableau, you've generated a random partition in the Plancherel measure. The permutation that resulted in the above partition of 50 is:

13,33,9,17,24,1,49,2,42,19,27,5,21,4,44,8,34,29,41,38,26,7,45,23,36,3,35,20,43,46,40,16,32,18,28,48,22,6,11,50,39,12,14,47,31,37,10,25,15,30

Because of this connection with permutations, the Plancherel measure is useful for studying the theory of random permutations, which can appear when studying the efficiency of sorting algorithms.

[1] In general, the measure is on irreducible unitary representations of a compact group G, but it's harder to put those on a pie.