A Diagram-Like Basis for the Multiset Partition Algebra

(Part of my thesis work under the supervision of Rosa Orellana)

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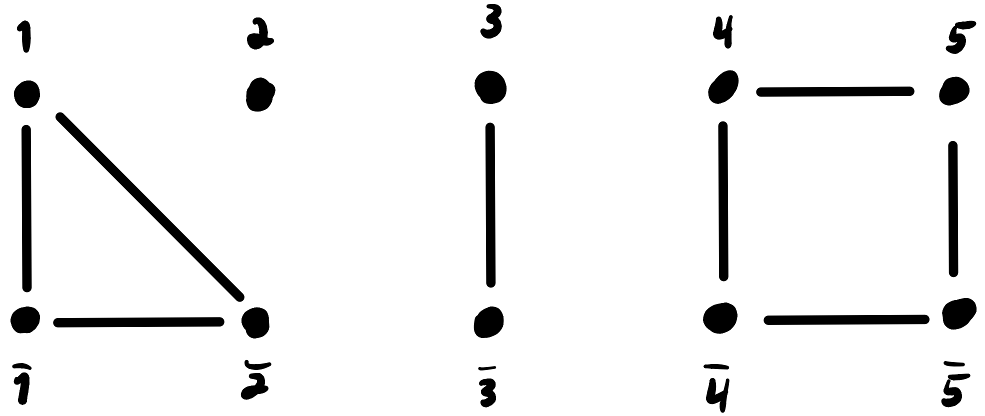
A Monoid Structure on Diagrams

An example of what we'll call a <u>partition diagram</u>:



<u>Key features</u>

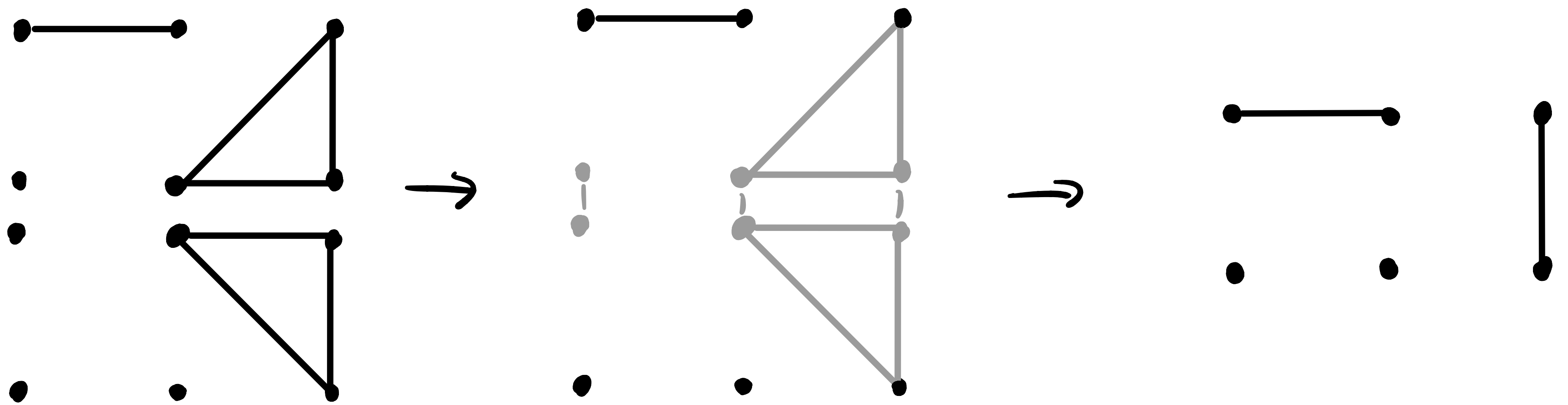
- Has r labeled vertices on top and bottom for some $r > 0$
- The vertices are grouped into connected components by edges.

A Monoid Structure on Diagrams

A multiplication formula:

i) Put the first diagram on top of the second, identifying the vertices in the middle

ii) Restrict to the top and bottom, preserving which vertices are connected in the larger diagram.



Schur-Weyl Duality

V_n : an n-dimensional $\mathbb{C}$-vector space

Gl_n : group of $n \times n$ invertible matrices over $\mathbb{C}$

$V_n^{\otimes r}$: the r^{th} tensor power of V_n. Think of elements as sequences

$$v_1 \otimes v_2 \otimes \cdots \otimes v_r$$

with each $v_i \in V_n$ (actually linear combinations of these)

Gl_n acts on $V_n^{\otimes r}$ in the following way

$$A \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_r) = (Av_1) \otimes (Av_2) \otimes \cdots \otimes (Av_r)$$

Schur-Weyl Duality

S_r : The symmetric group on r symbols

S_r also acts on $V_n^{\otimes r}$ by permuting tensor factors

$$\sigma . (v_1 \otimes v_2 \otimes \cdots \otimes v_r) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(r)}$$

$$GL_n \circlearrowright V_n^{\otimes r} \circlearrowleft S_r$$

Natural question: How do these actions interact with each other?

Schur-Weyl Duality

$$GL_n \circlearrowright V_n^{\otimes r} \circlearrowleft S_r$$

They are mutual centralizers

- $\mathrm{End}_{S_r}(V_n^{\otimes r})$ is generated by the GL_n-action

  Maps $V_n^{\otimes r} \to V_n^{\otimes r}$ which commute with the S_r-action

- $\mathrm{End}_{GL_n}(V_n^{\otimes r})$ is generated by the S_r-action

Schur-Weyl Duality

This is an example of Schur-Weyl duality, first discovered by Schur and then popularized by Weyl who used it to classify U_n and GL_n representations.

Main Takeaway:

This duality connects the representation theory of the two objects.

More precisely:

$$V_n^{\otimes r} \cong \bigoplus_{\lambda} E^{\lambda} \otimes S^{\lambda} \quad \text{as a } GL_n \times S_r\text{-module}$$

The Partition Algebra

We can restrict the GL_n action to the $n \times n$ permutation matrices

$$\begin{tikzcd}
GL_n \arrow[ddrr, dash] & & \color{teal}{\mathrm{End}_{S_n}(V_n^{\otimes r})} \arrow[ddll, dash] \\
& V_n^{\otimes r} & \\
S_n \arrow[uu, hook] & & S_r \arrow[uu, hook]
\end{tikzcd}$$

To get a sense for working with these centralizers, lets walk through this classical case.

The Partition Algebra

Given a basis $e_1, \dots, e_n$ of V_n, there is a basis of $V_n^{\otimes r}$ indexed by sequences $\underline{i} = (i_1, \dots, i_r) \in [n]^r$:

$$e_{\underline{i}} = e_{i_1} \otimes \cdots \otimes e_{i_r}$$

A permutation $\sigma \in S_n$ acts on $e_{\underline{i}}$ by:

$$\sigma . e_{\underline{i}} = (\sigma e_{i_1}) \otimes \cdots \otimes (\sigma e_{i_r})$$

$$= e_{\sigma(i_1)} \otimes \cdots \otimes e_{\sigma(i_r)}$$

$$= e_{\sigma(\underline{i})}$$

where $\sigma(\underline{i}) = (\sigma(i_1), \dots, \sigma(i_r))$

The Partition Algebra

Generally for $M \in \mathrm{End}(V_n^{\otimes r})$, we can describe it by its matrix coefficients relative to this basis:

$$M e_{\underline{i}} = \sum_{\underline{j}} M_{\underline{j}}^{\underline{i}} e_{\underline{j}}$$

The condition $M \in \mathrm{End}_{S_n}(V_n^{\otimes r})$ amounts to:

$$M_{\underline{j}}^{\underline{i}} = M_{\sigma(\underline{j})}^{\sigma(\underline{i})} \quad \text{for all } \underline{i}, \underline{j}, \sigma$$

The Partition Algebra

Visualizing some of these conditions for $\mathrm{End}_{S_3}(V_3^{\otimes 2})$:

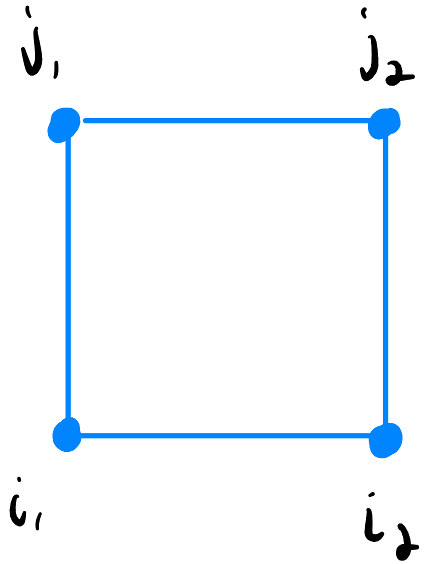
$$
\begin{array}{c|c}
 & j \\
i & \begin{array}{ccccccccc} 11 & 12 & 13 & 21 & 22 & 23 & 31 & 32 & 33 \end{array} \\
\begin{array}{c} 11 \\ 12 \\ 13 \\ 21 \\ 22 \\ 23 \\ 31 \\ 32 \\ 33 \end{array} &
\begin{bmatrix}
a & & & & & & & & \\
 & & b & & & & & & c \\
 & b & & & c & & & & \\
 & & & & & b & & & c \\
 & & & & a & & & & \\
c & & & b & & & & & \\
 & & & & c & & & b & \\
c & & & & & & b & & \\
 & & & & & & & & a
\end{bmatrix}
\end{array}
$$

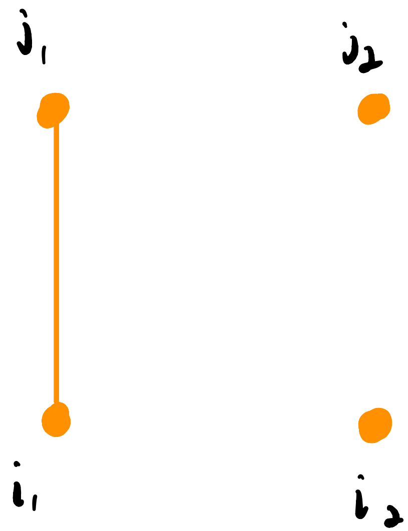
σ			
ε	(11, 11)	(12, 13)	(12, 33)
(12)	22, 22	21, 23	21, 33
(13)	33, 33	32, 31	32, 11
(23)	11, 11	13, 12	13, 22
(123)	22, 22	23, 21	23, 11
(132)	33, 33	31, 32	31, 22

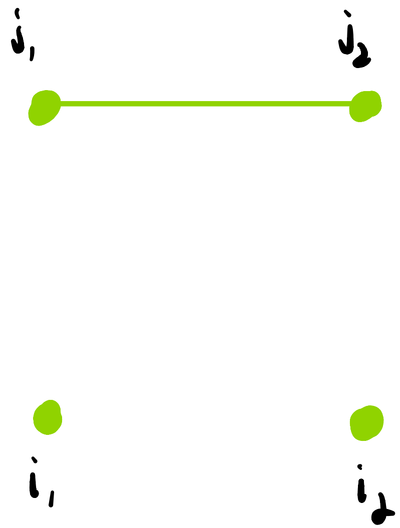
Each orbit represents a basis element, so how do we compactly represent each orbit?

The Partition Algebra

σ				
ε	$\underline{(11, 11)}$	$\underline{(12, 13)}$	$\underline{(12, 33)}$	$(i_1 i_2, j_1 j_2)$
(12)	$22, 22$	$21, 23$	$21, 33$	
(13)	$33, 33$	$32, 31$	$32, 11$	
(23)	$11, 11$	$13, 12$	$13, 22$	
(123)	$22, 22$	$23, 21$	$23, 11$	
(132)	$33, 33$	$31, 32$	$31, 22$	

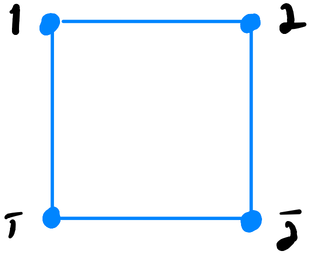




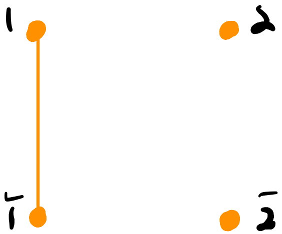


The Partition Algebra

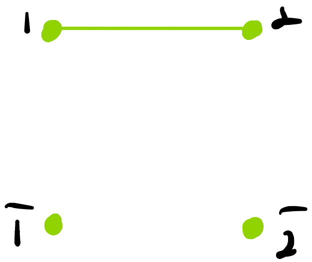
If we label these graphs with $1, \ldots, r$ on top and $\bar{1}, \ldots, \bar{r}$ on bottom, we get set partitions from connected components.



$$\longrightarrow \{\{1, 2, \bar{1}, \bar{2}\}\}$$



$$\longrightarrow \{\{1, \bar{1}\}, \{2\}, \{\bar{2}\}\}$$

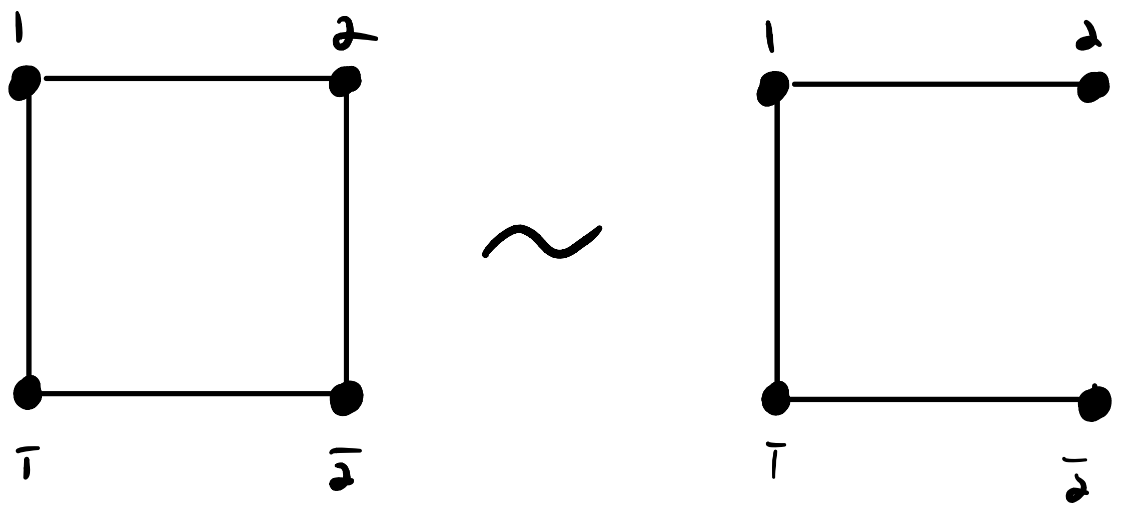


$$\longrightarrow \{\{1, 2\}, \{\bar{1}\}, \{\bar{2}\}\}$$

Write Π_{2r} for the set of set partitions of $[r] \cup [\bar{r}]$.

The Partition Algebra

These graphs representing orbits are not unique:

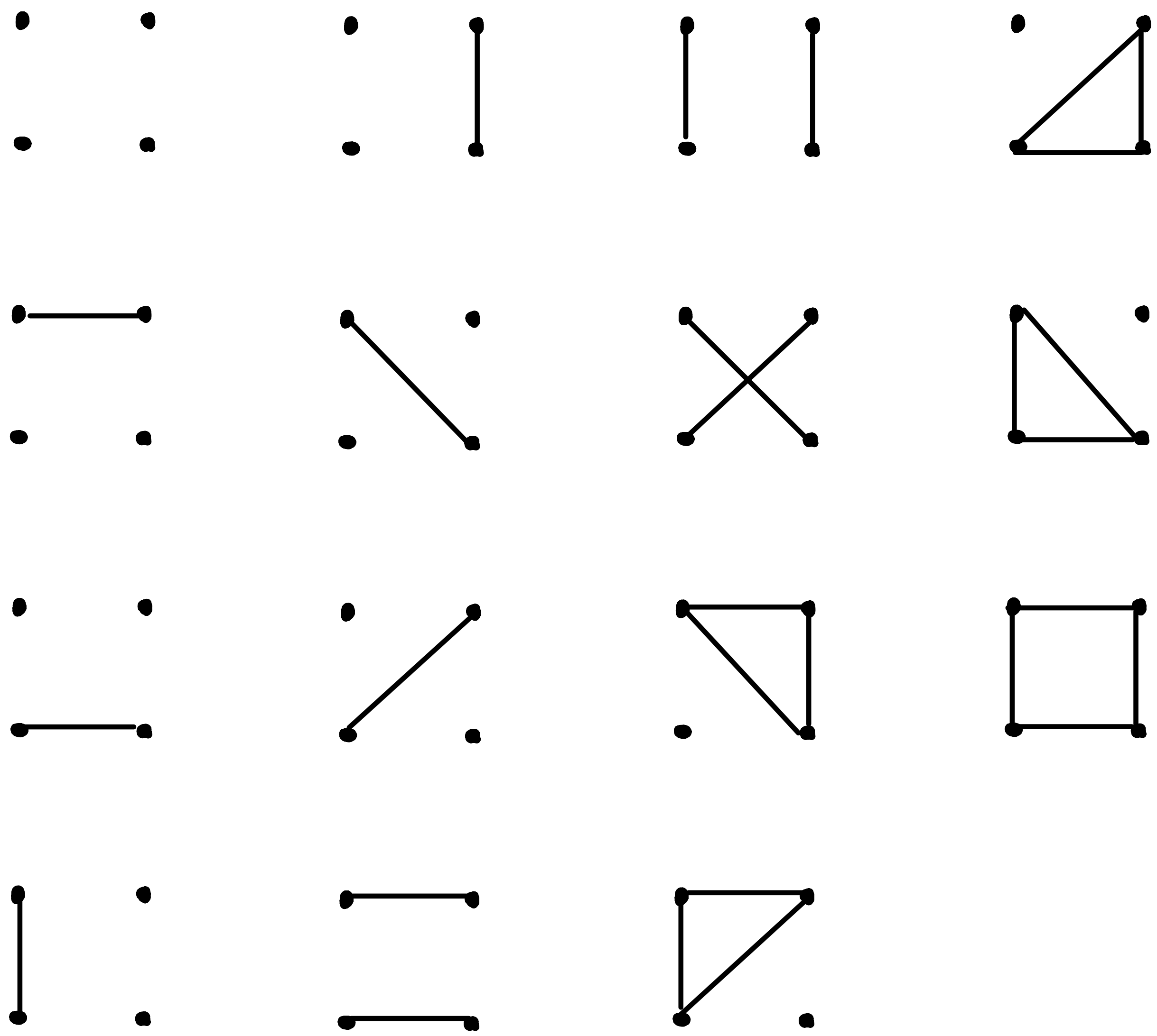


A **diagram** is an equivalence class of graphs on the vertices $[r] \cup [\bar{r}]$ with the same connected components

They are in correspondence with set partitions in Π_{2r}

The Partition Algebra

For example, $\mathrm{End}_{S_4}(V_4^{\otimes 2})$ has a basis indexed by:



(need $n \geq 2r$ for all the diagrams to appear)

The Partition Algebra

We'll now call $\mathrm{End}_{S_n}(V_n^{\otimes r})$ the Partition algebra $P_r(n)$ (introduced by Jones and by P. Martin in the 90s)

The basis obtained this way is called the orbit basis, which we'll write as

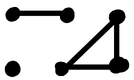
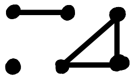
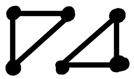
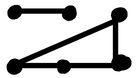
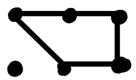
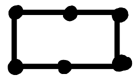
$$\{ T_\pi \ : \ \pi \in \Pi_{2r} \}$$

The Partition Algebra

There is another basis $\{L_{\pi}\}$ called the diagram basis given by:

$$L_{\pi} = \sum_{\nu \leq \pi} T_{\nu}$$

ν is a coarsening of π

Ex) L  $=$ T  $+$ T  $+$ T  $+$ T  $+$ T 

The Partition Algebra

Orbit basis example:

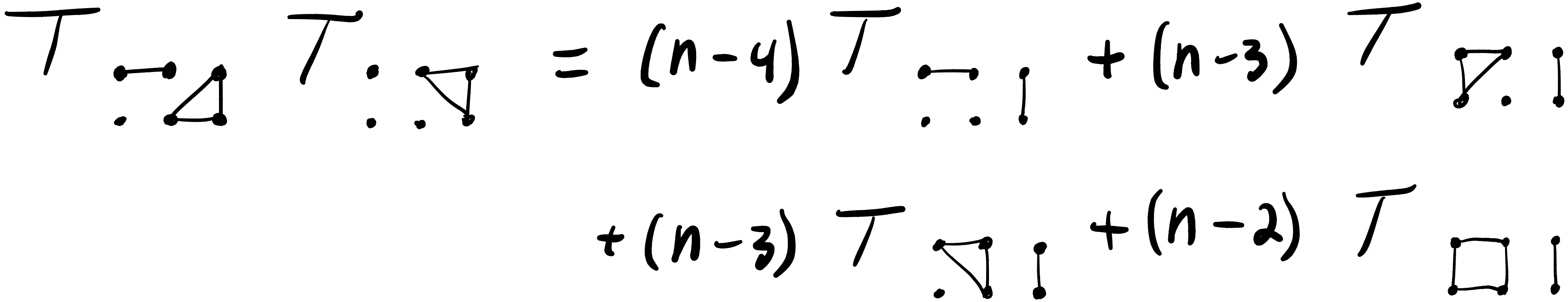
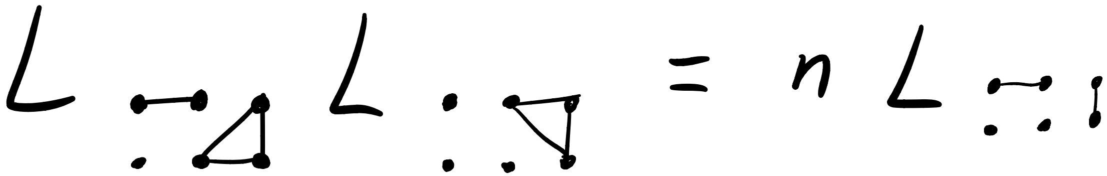


Diagram basis example:



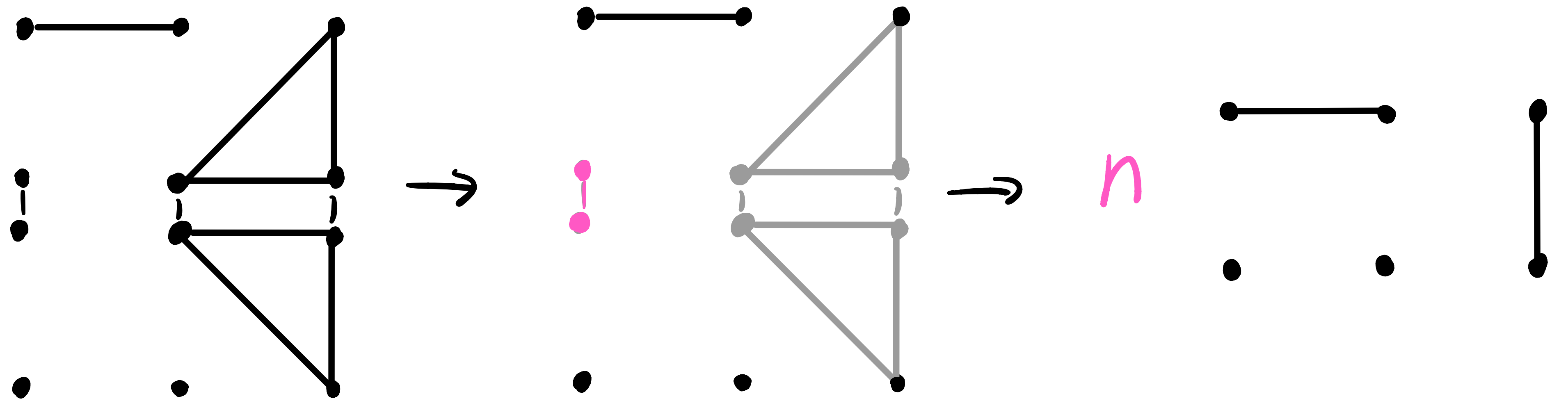
The Partition Algebra

The formula:

i) Put the first diagram on top of the second

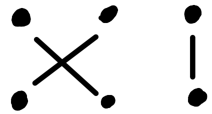
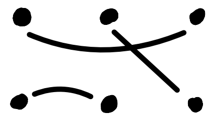
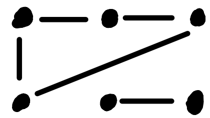
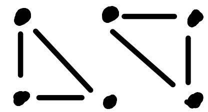
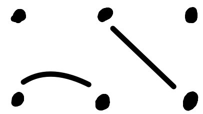
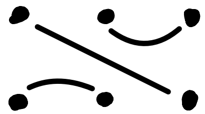
ii) Restrict to the top and bottom, preserving which vertices are connected in the larger diagram.

iii) Record a coefficient of n^c where c is the number of components stranded in the middle.



The Partition Algebra

$$G \circlearrowright V_n^{\otimes r} \circlearrowleft A$$

G	A	Typical Element	
GL_n	$\mathbb{C}S_r$		
O_n	Brauer Algebra ($B_r(n)$)		(matchings)
$G(m,p,n)$	Tanabe Algebra ($T_{m,p,r}(n)$)		(Subtle, but akin to $\#\text{top} \equiv \#\text{bottom} \pmod{m}$ for each block)
S_n	Partition Algebra		
$U_q(\mathfrak{gl}_2)$	Motzkin Algebra		(components of size ≤ 2, non-crossing)
$U_q(\mathfrak{sl}_2)$	Temperley-Lieb Algebra		(non-crossing matchings)

Howe Duality

$V_{n,k}$: The space of $n \times k$ matrices over $\mathbb{C}$

$P^r(V_{n,k})$: The space of homogeneous polynomial forms on $V_{n,k}$

These are homogeneous polynomials of degree r in indeterminates

$$x_{ij} \quad \text{for} \quad 1 \le i \le n, \quad 1 \le j \le k$$

where x_{ij} picks out the entry ij in the matrix:

$$x_{12}\, x_{13}\, x_{22}\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\right) = 2 \cdot 3 \cdot 5$$

Howe Duality

In the 1980s, Roger Howe determined that

$$GL_n \circlearrowright \mathcal{P}^r(V_{n,k}) \circlearrowleft GL_k$$

form a mutually centralizing pair where

- $A \in GL_n$ acts by $(A.f)(X) = f(A^{-1}X)$
- $B \in GL_k$ acts by $(B.f)(X) = f(XB)$

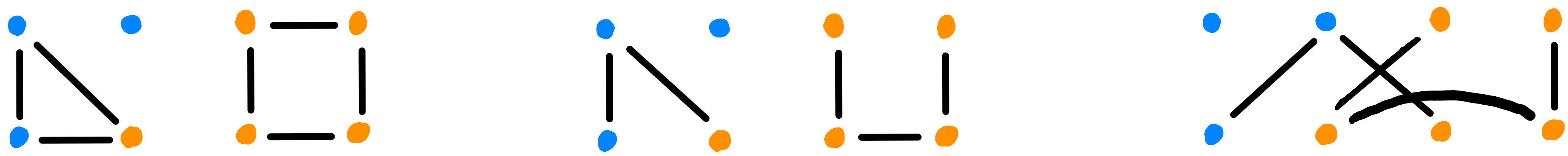
Howe Duality

$$
\begin{tikzcd}
GL_n \arrow[ddrr, dash] & P^r(V_{n,k}) & \mathrm{End}_{S_n}(P^r(V_{n,k})) \arrow[ddll, dash] \\
& & \\
S_n \arrow[uu, hook] & & GL_k \arrow[uu, hook]
\end{tikzcd}
$$

The Multiset Partition Algebra

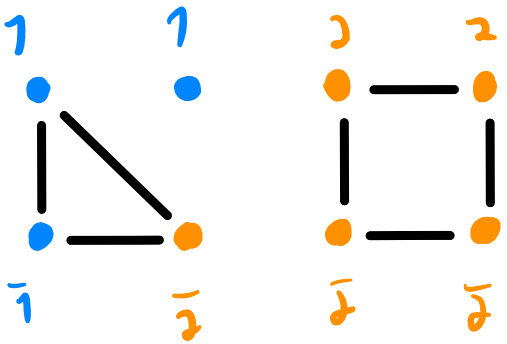
Orellana and Zabrocki (2020) examined $\mathrm{End}_{S_n}(P^r(V_{n,k}))$, describing an orbit basis for it and naming it $MP_{r,k}(n)$, the multiset partition algebra.

This basis is indexed by diagrams whose vertices are colored from a set of k colors with identically colored vertices among the top or bottom indistinguishable.



The Multiset Partition Algebra

These diagrams also represent partitions but now repitition is allowed (indicated by double brackets).



$$\longrightarrow \{\{ \{\{1,\bar{1},\bar{2}\}\}, \{\{1\}\}, \{\{2,2,\bar{2},\bar{2}\}\} \}\}$$

$1\bar{1}\bar{2}/2$ $\quad$ $1/2\bar{1}\bar{2}$

We'll write $\tilde{\Pi}_{2r,k}$ for the set of multiset partitions with r entries each from $[k]$ and $[\bar{k}]$.

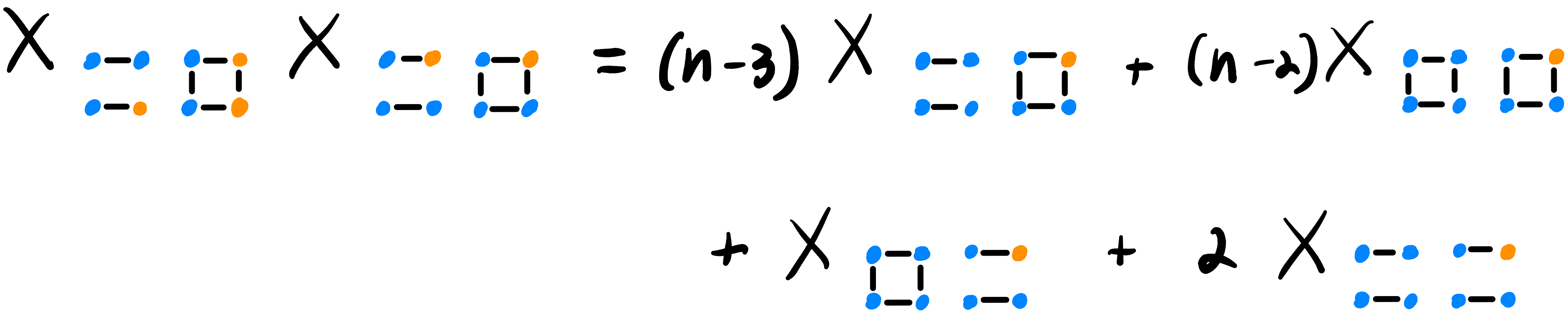
We will consistently use these colors:

1 = 

2 = 

The Multiset Partition Algebra

Writing $\{X_{\tilde{\pi}} : \tilde{\pi} \in \tilde{\Pi}_{2r,k}\}$ for the orbit basis obtained by Orellana and Zabrocki, an example of its multiplication is:



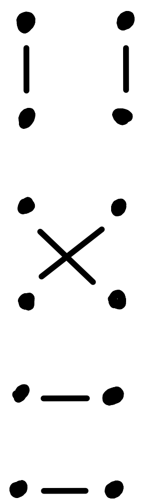
This looks like the orbit basis for $P_r(n)$. Can we change to a basis like the diagram basis?

The Multiset Partition Algebra

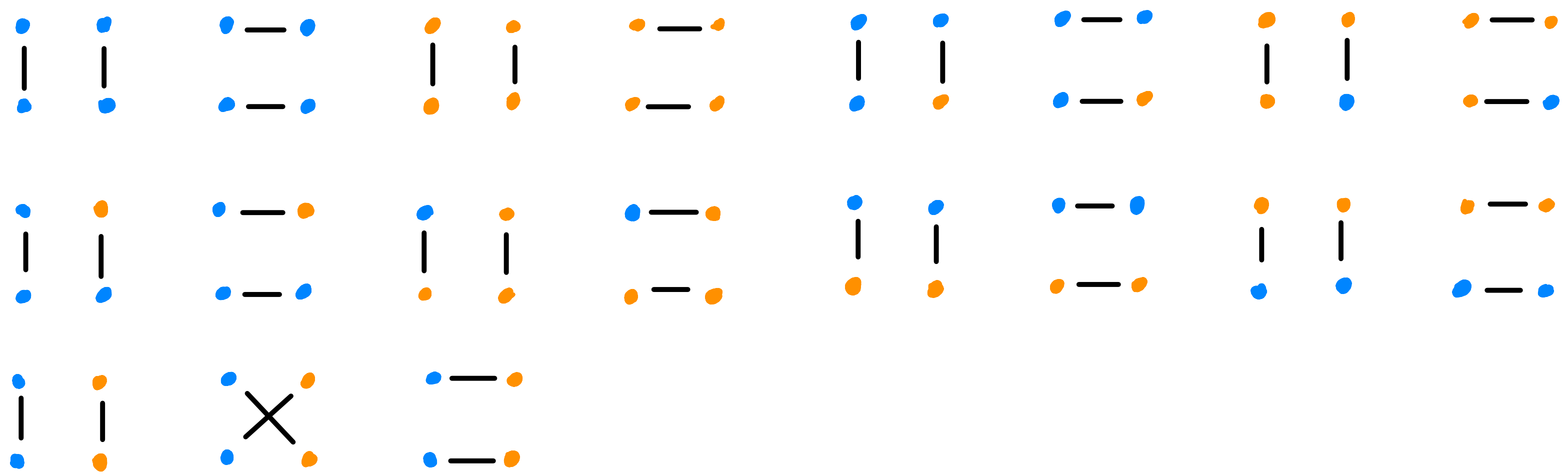
Let $S_r \subseteq A_r(n) \subseteq P_r(n)$ and define a new algebra $\tilde{A}_{r,k}(n)$ called the corresponding Painted algebra with basis:

$$\left\{ D_{\tilde{\pi}} : \begin{array}{c} \tilde{\pi} \text{ obtained by coloring the vertices} \\ \text{of a diagram in } A_r(n) \end{array} \right\}$$

$B_2(n)$

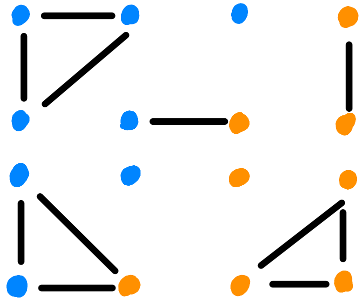


$\tilde{B}_{2,2}(n)$



The Multiset Partition Algebra

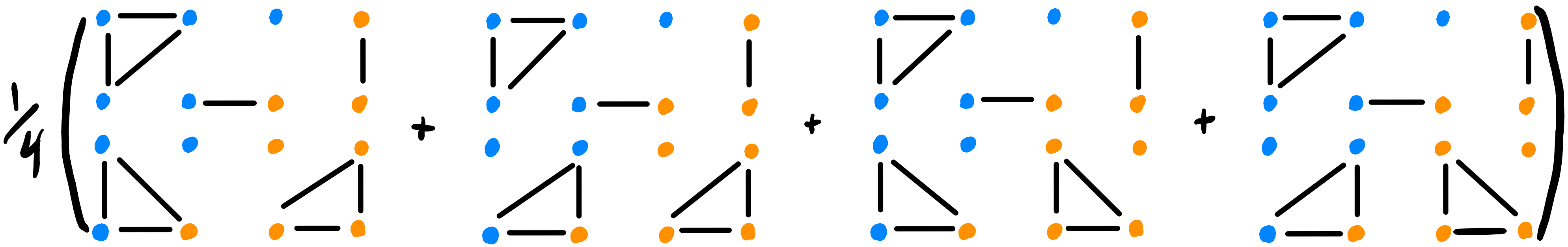
The product is given by:



Colors must match in middle or else product is zero

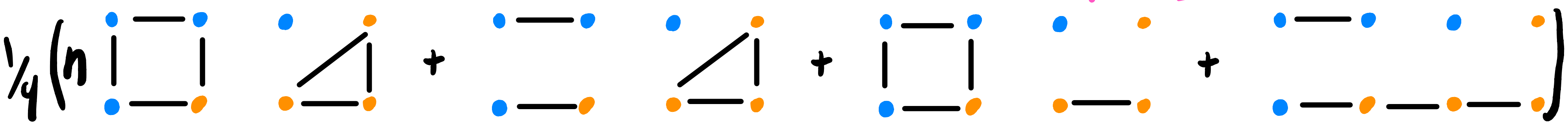
↓

Average over permutations of the top of the second diagram



↓

Take the product as in $P_r(n)$



The Multiset Partition Algebra

Theorem Let $S_n \subseteq G \subseteq GL_n$ be a subgroup with

$$\mathrm{End}_G(V_n^{\otimes r}) = A_r(n).$$

Then

$$\mathrm{End}_G(P^r(V_{n,\kappa})) = \tilde{A}_{r,\kappa}(n).$$

Corollary $MP_{r,\kappa}(n) \cong \tilde{P}_{r,\kappa}(n)$. We call the basis $\{D_{\tilde{\pi}}\}$ of $MP_{r,\kappa}(n)$ the diagram-like basis

Subalgebras

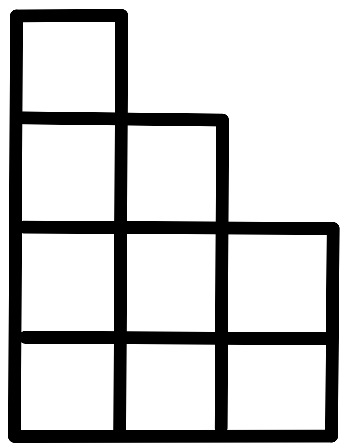
$$
\begin{tikzcd}
P_r(n) & & GL_n & & MP_{r,k}(n) \\
& V_n^{\otimes r} & & P^r(V_{n,k}) & \\
A_r(n) \arrow[uu, hook] & & G \arrow[uu, hook] & & \tilde{A}_{r,k}(n) \arrow[uu, hook] \\
& & & & \\
S_r \arrow[uu, hook] & & S_n \arrow[uu, hook] & & GL_k \arrow[uu, hook]
\end{tikzcd}
$$

Representations

A **partition** λ of n is a weakly decreasing sequence $(\lambda_1, \ldots, \lambda_\ell)$ of positive integers which sum to n.

The **Young diagram** of λ is an array of left-justified boxes with λ_i boxes in the i^{th} row from the bottom.

$(3, 3, 2, 1) \longleftrightarrow$

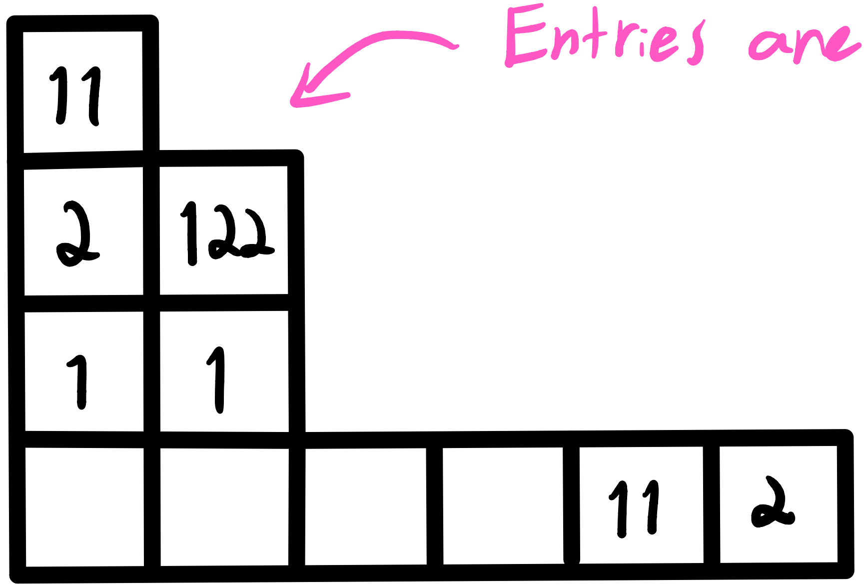


Representations

A Multiset Partition tableau of shape λ is a filling of λ's Young diagram like so:

Only the first row has empty boxes, at least as many as λ_2

Entries are multisets



Write $MSPT(\lambda, r, k)$ for the set of these tableaux with a total of r numbers from $[k]$.

Representations

Order multisets by the last-letter order:

$11 < 2 \qquad 12 < 22 \qquad 22 < 122$

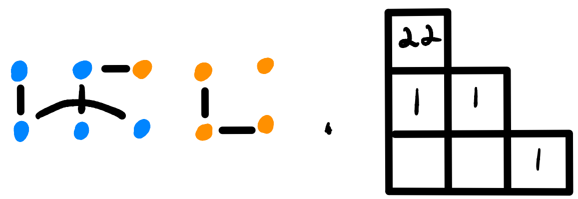
A semistandard multiset partition tableau has rows weakly increasing and columns strictly increasing.

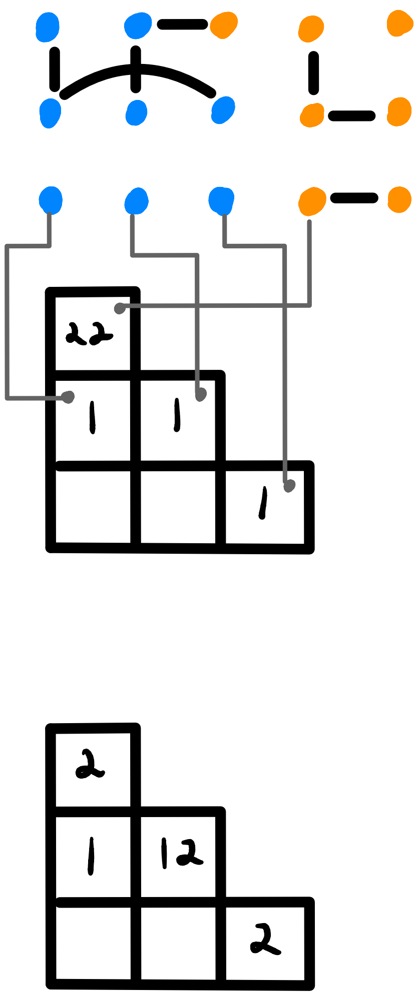
22					
2	12				
1	1				
				11	2

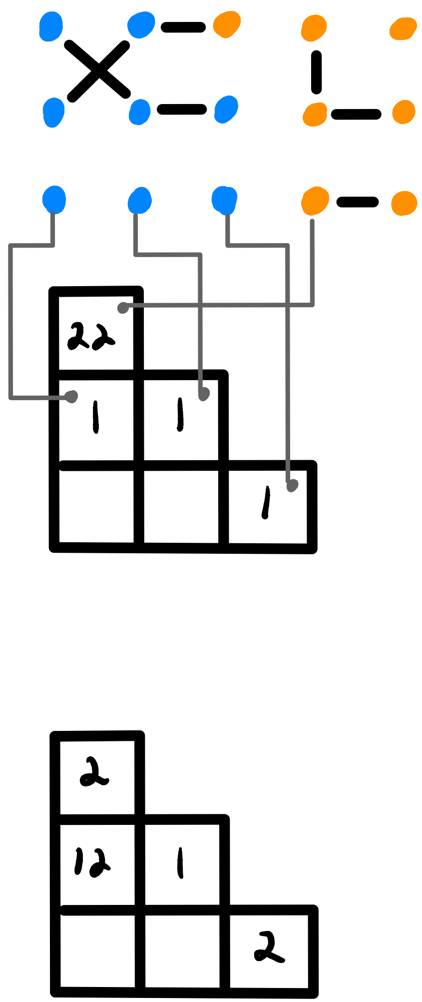
Write $SSMPT(\lambda, r, k)$ for the set of these.

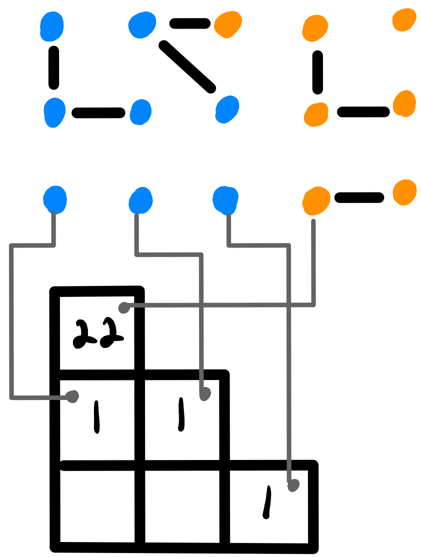
Representations

An example of the action:



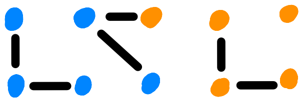






X Two blocks above the first row get combined

Representations

 $\cdot$ $\begin{array}{|c|c|c|} \cline{1-1} 22 \\ \cline{1-2} 1 & 1 \\ \hline & & 1 \\ \hline \end{array} = \frac{1}{3}\left(\begin{array}{|c|c|c|} \cline{1-1} 2 \\ \cline{1-2} 1 & 12 \\ \hline & & 2 \\ \hline \end{array} + \begin{array}{|c|c|c|} \cline{1-1} 2 \\ \cline{1-2} 12 & 1 \\ \hline & & 2 \\ \hline \end{array} \right)$

Straightening algorithm

$$= \frac{1}{3}\left(\begin{array}{|c|c|c|} \cline{1-1} 2 \\ \cline{1-2} 1 & 12 \\ \hline & & 2 \\ \hline \end{array} + \begin{array}{|c|c|c|} \cline{1-1} 12 \\ \cline{1-2} 1 & 2 \\ \hline & & 2 \\ \hline \end{array} - \begin{array}{|c|c|c|} \cline{1-1} 2 \\ \cline{1-2} 1 & 12 \\ \hline & & 2 \\ \hline \end{array} \right)$$

$MP^{\lambda}_{r,k}$:= Span of $SSMPT(\lambda, r, k)$

Theorem The $MP^{\lambda}_{r,k}$ for $\lambda \vdash n$ and $\sum_{i=2}^{\ell(\lambda)} \lceil \frac{i-1}{k} \rceil \lambda_i$ form a complete set of irreducible representations for $MP_{r,k}(n)$ when $n \geq 2r$.

Proof Sketch

Break $P^r(V_{n,k})$ into pieces $\mathcal{U}_{\underline{a}}$ based on the second index.

E.g. $x_{11}x_{21}x_{22}x_{22} \in \mathcal{U}_{(2,2)}$

Write $W_{r,k}$ for weak compositions of r of length k

Then $P^r(V_{n,k}) \cong \bigoplus_{\underline{a} \in W_{r,k}} \mathcal{U}_{\underline{a}}$ as a Gl_n-module

Proof Sketch

$S_{\underline{a}}$: Young subgroup $\qquad s_{\underline{a}} = \frac{1}{|S_{\underline{a}}|} \sum_{\sigma \in S_{\underline{a}}} \sigma$

E.g. $\quad S_{(2,2)} = S_{\{1,2\}} \times S_{\{3,4\}}$

$$s_{(2,2)} = \frac{1}{4}\left(1234 + 2134 + 1243 + 2143\right)$$

Recall S_r acts on $V_n^{\otimes r}$ by permuting factors

$$s_{(2,2)}(e_1 \otimes e_2 \otimes e_2 \otimes e_2) = \frac{1}{2}\left(e_1 \otimes e_2 \otimes e_2 \otimes e_2 + e_2 \otimes e_1 \otimes e_2 \otimes e_2\right)$$

Proof Sketch

As vector spaces,

$$\Phi : \mathcal{U}_{\underline{a}} \xrightarrow{\sim} s_{\underline{a}} V_n^{\otimes r}$$

$$x_{11} x_{21} x_{22} x_{22} \longmapsto s_{(2,2)} (e_1 \otimes e_2 \otimes e_2 \otimes e_2)$$

They both have a GL_n-action but are not clearly isomorphic as GL_n-modules.

For $M \in GL_n$,

$$\Phi M = M^{-1} \Phi$$

Proof Sketch

However, we get an induced isomorphism

$$\mathrm{End}_G\Big(\bigoplus_{\underline{a}\in W_{r,n}} \mathcal{U}_{\underline{a}}\Big) \cong \mathrm{End}_G\Big(\bigoplus_{\underline{a}\in W_{r,n}} s_{\underline{a}} V_n^{\otimes r}\Big)$$

$$\psi \longmapsto \Phi \circ \psi \circ \Phi^{-1}$$

Note for $M \in GL_n$,

$$\Phi \psi \Phi^{-1} M = \Phi \psi M^{-1} \Phi^{-1} = \Phi M^{-1} \psi \Phi^{-1} = M \Phi \psi \Phi^{-1}$$

Proof Sketch

$$\mathrm{End}_G\left(\mathcal{P}^r(V_{n,u})\right) \cong \mathrm{End}_G\left(\bigoplus_{\underline{a}} s_{\underline{a}} V_n^{\otimes r}\right)$$

$$\cong \bigoplus_{\underline{a},\underline{b}} \mathrm{Hom}_G\left(s_{\underline{b}} V_n^{\otimes r}, s_{\underline{a}} V_n^{\otimes r}\right)$$

$$\cong \bigoplus_{\underline{a},\underline{b}} s_{\underline{a}}\, \mathrm{End}_G\left(V_n^{\otimes r}\right) s_{\underline{b}}$$

with product

$$(s_{\underline{a}}\, \pi\, s_{\underline{b}}) \cdot (s_{\underline{c}}\, \gamma\, s_{\underline{d}}) = \delta_{\underline{b},\underline{c}} \left(s_{\underline{a}}\, \pi\, s_{\underline{b}}\, \gamma\, s_{\underline{d}}\right)$$

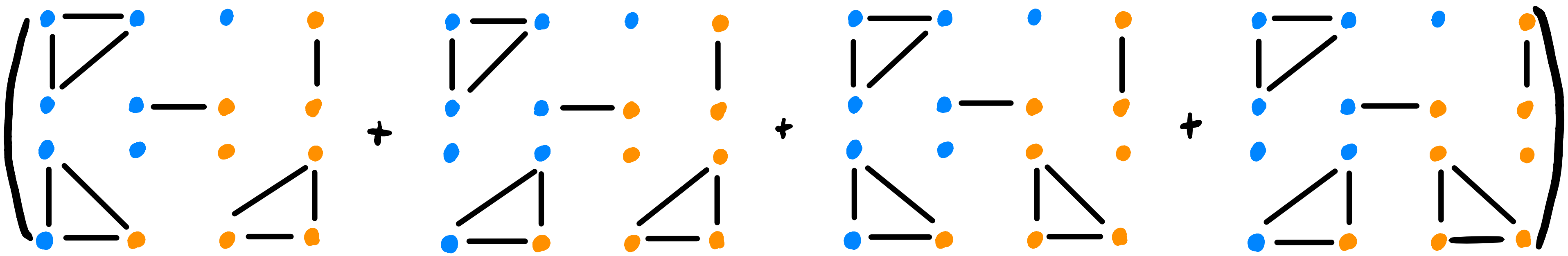
The Multiset Partition Algebra

Colors match in middle

$$(s_{\underline{a}}\, \pi\, s_{\underline{b}}) \cdot (s_{\underline{c}}\, \gamma\, s_{\underline{d}}) = \delta_{\underline{b},\underline{c}} \left(s_{\underline{a}}\, \pi\, s_{\underline{b}}\, \gamma\, s_{\underline{d}} \right)$$

$$= \delta_{\underline{b},\underline{c}} \frac{1}{|S_{\underline{b}}|} \sum_{\sigma \in S_{\underline{b}}} s_{\underline{a}}\, \pi\, \sigma\, \gamma\, s_{\underline{d}}$$

permutations of top of the second diagram

$\frac{1}{4}$ 

Proof Sketch

Proof Summary

- Decompose $P^r(V_{n,k})$
- Leads to a decomposition of $End_G(P^r(V_{n,k}))$ via idempotents
- The diagram-like basis comes from sandwiching an idempotent between two partition diagrams

Thank you!