Super Multiset RSK and a Mixed Multiset Partition Algebra

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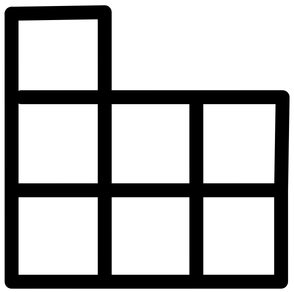
Part one:

Crash course in Representations of S_n

Tableau Model for Symmetric Group Representations

- A Young diagram is an array of boxes justified to the left and below.
- A partition $\lambda \vdash n$ is a weakly decreasing sequence of positive integers summing to n.
- The shape of a Young diagram is the sequence of its row lengths from bottom to top.

Ex1



$$\lambda = (3, 3, 1)$$

Tableau Model for Symmetric Group Representations

- A standard Young tableau of shape $\lambda \vdash n$ is a filling of the Young diagram with the numbers $1, \ldots, n$ so that rows and columns are increasing

Ex 1

$$\begin{array}{|c|c|c|}
\hline
4 & \multicolumn{2}{c}{} \\
\hline
2 & 6 & 7 \\
\hline
1 & 3 & 5 \\
\hline
\end{array}$$

- Write $SYT(\lambda)$ for the set of Standard Young tableaux of shape λ.

Tableau Model for Symmetric Group Representations

Have S_n act on $SYT(\lambda)$ in the natural way:

EX

$$(123) \cdot \begin{array}{|c|c} \hline 2 & \multicolumn{1}{|c}{} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} = \begin{array}{|c|c} \hline 3 & \multicolumn{1}{|c}{} \\ \hline 2 & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

... but this no longer standard!

Tableau Model for Symmetric Group Representations

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... but this no longer standard!

The straightening algorithm rewrites a nonstandard tableau as a linear combination of standard ones.

$$\begin{array}{|c|c} \hline 3 & \multicolumn{1}{|c}{} \\ \hline 2 & \multicolumn{1}{c|}{1} \\ \hline \end{array} = \begin{array}{|c|c} \hline 2 & \multicolumn{1}{|c}{} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} - \begin{array}{|c|c} \hline 3 & \multicolumn{1}{|c}{} \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

RSK Algorithm

- A general representation theory fact:

$$\mathbb{C}S_n \cong \bigoplus_{\lambda \vdash n} S^\lambda \otimes S^\lambda$$

as an $S_n \times S_n$-representation.

- Comparing dimensions,

$$n! = \sum_{\lambda \vdash n} |SYT(\lambda)|^2$$

- Suggests a bijection

$$S_n \xleftrightarrow{\sim} \biguplus_{\lambda \vdash n} SYT(\lambda) \times SYT(\lambda)$$

RSK Algorithm

Ex)

$$1\ 2\ 3 \qquad \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \ , \ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \right)$$

$$1\ 3\ 2 \qquad \left(\begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \ , \ \begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \right)$$

$$2\ 1\ 3 \qquad \left(\begin{array}{|c|c}\cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \ , \ \begin{array}{|c|c}\cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \right)$$

$$2\ 3\ 1 \qquad \left(\begin{array}{|c|c}\cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \ , \ \begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \right)$$

$$3\ 1\ 2 \qquad \left(\begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \ , \ \begin{array}{|c|c}\cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \right)$$

$$3\ 2\ 1 \qquad \left(\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \ , \ \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \right)$$

Part Two:

Centralizer Algebras

Schur-Weyl Duality

V_n : an n-dimensional $\mathbb{C}$-vector space

GL_n : The group of $n \times n$ invertible matrices over $\mathbb{C}$

$V_n^{\otimes r}$: the r^{th} tensor power of V_n. Think of elements as sequences

$$v_1 \otimes v_2 \otimes \cdots \otimes v_r$$

with each $v_i \in V_n$ (actually linear combinations of these)

GL_n acts on $V_n^{\otimes r}$ in the following way

$$A.(v_1 \otimes v_2 \otimes \cdots \otimes v_r) = (Av_1) \otimes (Av_2) \otimes \cdots \otimes (Av_r)$$

Schur-Weyl Duality

S_r also acts on $V_n^{\otimes r}$ by permuting tensor factors

$$\sigma . (v_1 \otimes v_2 \otimes \cdots \otimes v_r) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(r)}$$

$$GL_n \circlearrowright V_n^{\otimes r} \circlearrowleft S_r$$

Natural question: How do these actions interact with each other?

Schur-Weyl Duality

$$GL_n \circlearrowright V_n^{\otimes r} \circlearrowleft S_r$$

They are mutual centralizers

- $\mathrm{End}_{S_r}(V_n^{\otimes r})$ is generated by the GL_n-action

  Maps $V_n^{\otimes r} \to V_n^{\otimes r}$ which commute with the S_r-action

- $\mathrm{End}_{GL_n}(V_n^{\otimes r})$ is generated by the S_r-action

Schur-Weyl Duality

This is an example of Schur-Weyl duality, first discovered by Schur and then popularized by Weyl who used it to classify U_n and GL_n representations.

Main Takeaway:

This duality connects the representation theory of the two objects, pairing up their irreducible representations.

More precisely:

$$V_n^{\otimes r} \cong \bigoplus_{\lambda} GL^{\lambda} \otimes S^{\lambda} \quad \text{as a } GL_n \times S_r\text{-module}$$

The Partition Algebra

We can restrict the GL_n action to the $n \times n$ permutation matrices

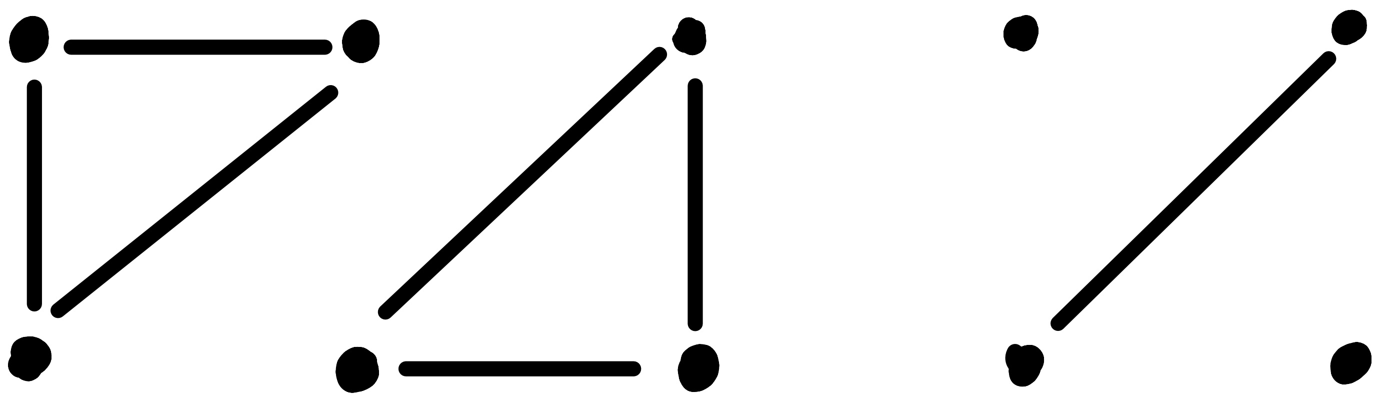
$$\begin{tikzcd}
GL_n \arrow[dr, dash] & & \mathrm{End}_{S_n}(V_n^{\otimes r}) = P_r(n) \arrow[dl, dash] \\
& V_n^{\otimes r} \arrow[dl, dash] \arrow[dr, dash] & \\
S_n \arrow[uu, hook] & & S_r \arrow[uu, hook]
\end{tikzcd}$$

The Partition Algebra

Elements of $P_r(n)$ can be described by

Partition diagrams

Ex



The Partition Algebra

Ordering sets by their largest element, we define a **Standard set partition tableau** as a set-valued tableaux with increasing rows and columns with at least λ_2 empty boxes in the first row

Ex]

15			
24	7	68	
			3

Write $SPT(\lambda)$ for the set of such tableaux of shape $\lambda \vdash n$ with maximum entry n.

The irreducible representation P_r^λ has a basis indexed by $SPT(\lambda)$.

RSK for the Partition Algebra

Call $P_r(n) = \mathrm{End}_{S_n}(V_n^{\otimes r})$ the partition algebra when $n \geq 2r$.

The decomposition

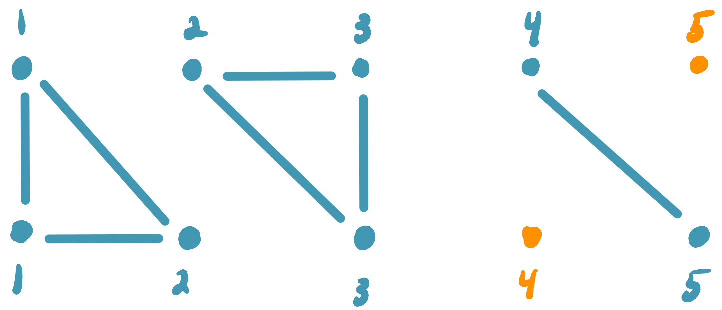
$$P_r(n) \cong \bigoplus_{\lambda \vdash n} P_r^{\lambda} \otimes P_r^{\lambda}$$

Suggests a bijection between

$$\left\{ \begin{array}{l} \text{partition diagrams} \\ \text{on } 2r \text{ vertices} \end{array} \right\} \overset{\sim}{\longleftrightarrow} \biguplus_{\lambda \vdash n} SPT(\lambda) \times SPT(\lambda)$$

RSK for the Partition Algebra

An RSK variant introduced in COSSZ20:

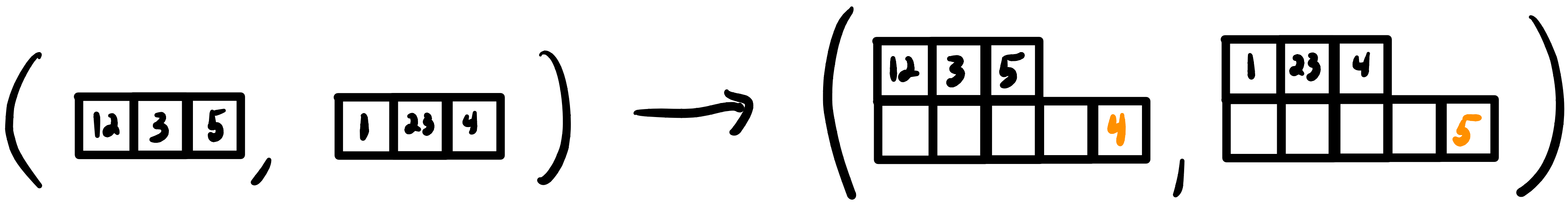


$$\downarrow$$

$$\begin{pmatrix} 1 & 23 & 4 \\ 12 & 3 & 5 \end{pmatrix}$$

← ordered by first row

$$\big\downarrow \text{RSK}$$



Recap of part two

- The centralizer algebra of a group acting on a vector space can tell you more about the group's representations.

- $P_r(n) = \mathrm{End}_{S_n}(V_n^{\otimes r})$ has a nice description in terms of partition diagrams.

- The irreducible representations P_r^λ have a description in terms of set-valued tableaux.

Part three:

Mixed Multiset Partition Algebra

The Mixed Multiset Partition Algebra

$\mathrm{Sym}^r(V_n)$: The r^{th} symmetric power of V_n

Typical element: $e_1 e_1 e_2 e_4 = e_1 e_1 e_4 e_2 = e_1 e_2 e_1 e_4 = \ldots,$

$\Lambda^r(V_n)$: The r^{th} exterior power of V_n

Typical element: $e_1 \wedge e_2 \wedge e_4 = - e_2 \wedge e_1 \wedge e_4 = \ldots,$

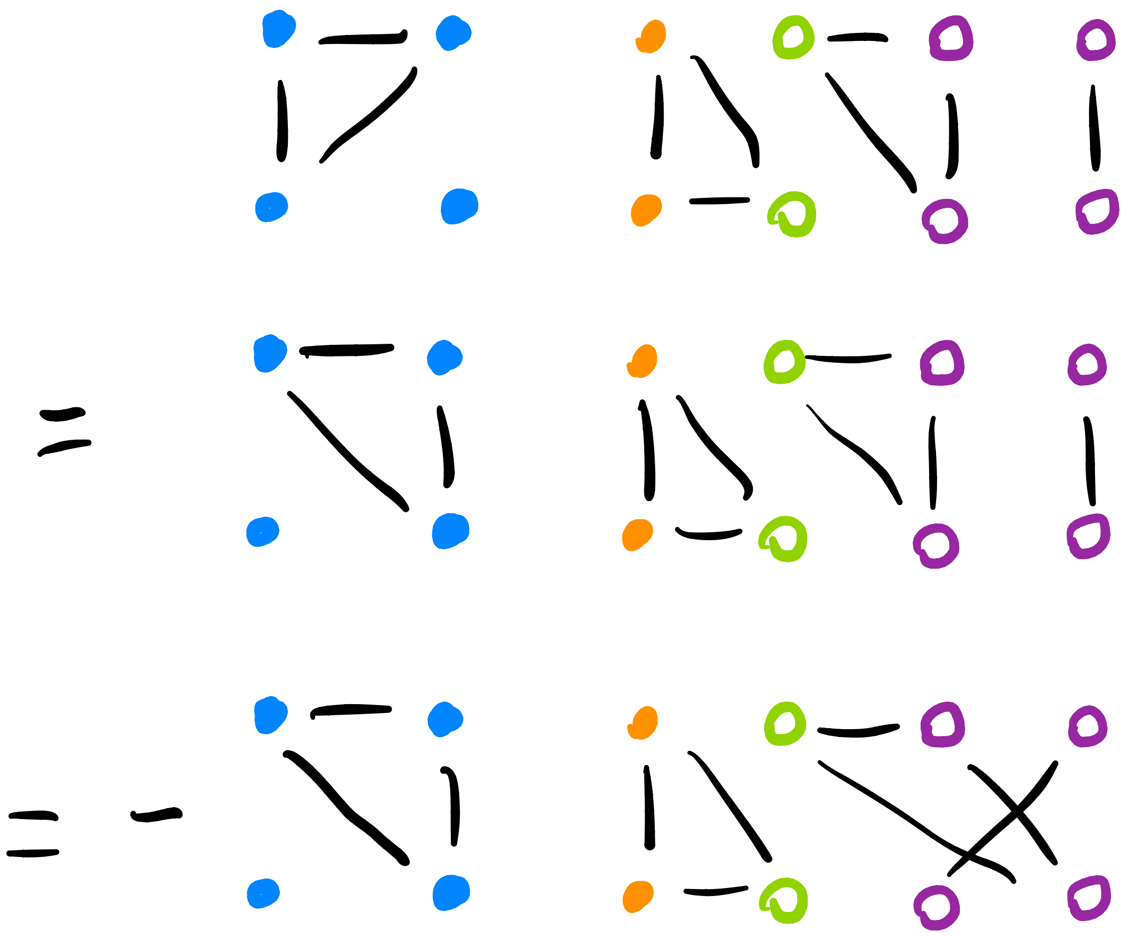
- Let $W = \mathrm{Sym}^{\underline{a}}(V_n) \otimes \Lambda^{\underline{b}}(V_n)$

$$= \mathrm{Sym}^{a_1}(V_n) \otimes \cdots \otimes \mathrm{Sym}^{a_k}(V_n) \otimes \Lambda^{b_1}(V_n) \otimes \cdots \otimes \Lambda^{b_\ell}(V_n)$$

- Interested in $MP_{\underline{a},\underline{b}}(n) = \mathrm{End}_{S_n}(W)$.

The Mixed Multiset Partition Algebra

$MP_{\underline{a},\underline{b}}(n)$ has a basis indexed by multiset partition diagrams $\Pi_{\underline{a},\underline{b}}$:



$\in \Pi_{(2,1),(1,2)}$

The Mixed Multiset Partition Algebra

- Now we want to think about multisets with elements

$$1 < 2 < 3 < \dots < \bar{1} < \bar{2} < \bar{3} < \dots$$

where barred numbers cannot be repeated

Ex)

$$\{\{1, 1, 2\}\} < \{\{1, \bar{3}\}\}$$

$$\{\{2, \bar{3}, \bar{4}\}\} < \{\{\bar{5}\}\}$$

order by largest element

The Mixed Multiset Partition Algebra

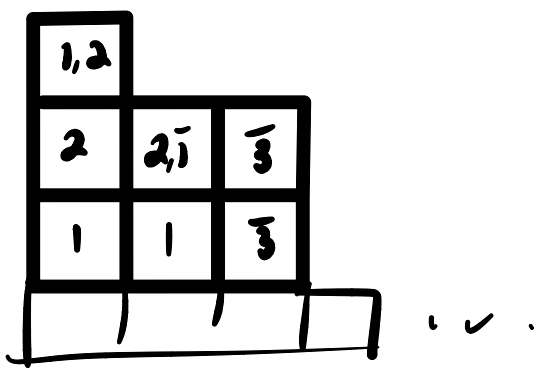
A **semistandard multiset partition tableau** is a filling of a Young diagram by multisets which:

i) Increases weakly along rows and up columns.

ii) Multisets with an <u>even</u> number of barred values can't repeat within a column.

iii) Multisets with an <u>odd</u> number of barred values can't repeat within a row.

Ex]



Write $SSMT(\lambda, \underline{a}, \underline{b})$ for the set of these tableaux of shape λ and multiplicities given by $\underline{a}$ and $\underline{b}$.

The Mixed Multiset Partition Algebra

- It's straightforward to find a <u>spanning</u> set of $MP^{\lambda}_{\underline{a},\underline{b}}$ indexed by $SSMT(\lambda, \underline{a}, \underline{b})$, so

$$\dim\left(MP^{\lambda}_{\underline{a},\underline{b}}\right) \leq |SSMT(\lambda, \underline{a}, \underline{b})|.$$

By representation theory facts,

$$|\Pi_{\underline{a},\underline{b}}| = \dim\left(MP_{\underline{a},\underline{b}}(n)\right) = \sum_{\lambda} \dim\left(MP^{\lambda}_{\underline{a},\underline{b}}\right)^2.$$

The Mixed Multiset Partition Algebra

If we had an RSK-like bijection

$$\Pi_{\underline{a},\underline{b}} \xleftrightarrow{\sim} \biguplus_{\lambda} SSMT(\lambda, \underline{a}, \underline{b}) \times SSMT(\lambda, \underline{a}, \underline{b}),$$

then

$$\sum_{\lambda} \dim(MP^{\lambda}_{\underline{a},\underline{b}})^2 = \sum_{\lambda} |SSMT(\lambda, \underline{a}, \underline{b})|^2.$$

Because $\dim(MP^{\lambda}_{\underline{a},\underline{b}}) \leq |SSMT(\lambda, \underline{a}, \underline{b})|$, we could conclude that

$$\dim(MP^{\lambda}_{\underline{a},\underline{b}}) = |SSMT(\lambda, \underline{a}, \underline{b})|.$$

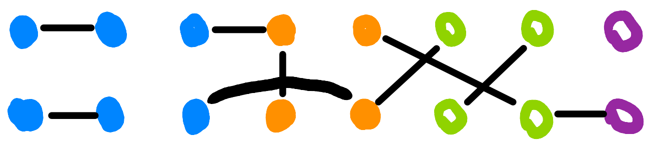
Super RSK

Super RSK (Muth 19) generalizes RSK to an alphabet with a $\mathbb{Z}/2\mathbb{Z}$-grading (i.e. labeled as even or odd)

It produces tableaux with weakly increasing rows/columns where

- even elements can't repeat in a column
- odd elements can't repeat in a row

Super Multiset RSK



$$\downarrow$$

$$\begin{pmatrix} 2 & 12 & \bar{1} & \bar{1} \\ \bar{1}\bar{2} & 2 & \bar{1} & 12 \end{pmatrix} \leftarrow \text{ordered by first row}$$

$\downarrow$ sRSK, treating a multiset as even iff it has an even number of barred elements

$$\left(\begin{array}{|c|c|} \hline 12 & \bar{1}\bar{2} \\ \hline 2 & \bar{1} \\ \hline \end{array} \,,\; \begin{array}{|c|c|} \hline 12 & \bar{1} \\ \hline 2 & \bar{1} \\ \hline \end{array} \right) \longrightarrow \left(\begin{array}{|c|c|cc} \hline 12 & \bar{1}\bar{2} & \multicolumn{2}{c}{} \\ \hline 2 & \bar{1} & \multicolumn{2}{c}{} \\ \hline & & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{11} \\ \hline \end{array} \,,\; \begin{array}{|c|c|cc} \hline 12 & \bar{1} & \multicolumn{2}{c}{} \\ \hline 2 & \bar{1} & \multicolumn{2}{c}{} \\ \hline & & \multicolumn{1}{c|}{11} & \multicolumn{1}{c|}{\bar{2}} \\ \hline \end{array} \right)$$

Recap of Part three

- The centralizer algebra of S_n acting on symmetric and exterior powers has a description in terms of multiset partition diagrams

- A generalization of RSK can be used to prove dimensions of irreducible representations.

Closing Remarks:

Symmetric Function Identities

Symmetric Function Identities

GL_n-module V	Character χ_V
$\mathrm{Sym}^{\underline{a}}(V_n)$	$h_{\underline{a}}(x_n)$
$\Lambda^{\underline{b}}(V_n)$	$e_{\underline{b}}(x_n)$
$\mathrm{Sym}^{\underline{a}}(V_n) \otimes \Lambda^{\underline{b}}(V_n)$	$h_{\underline{a}}(x_n)\, e_{\underline{b}}(x_n)$

Symmetric Function Identities

Example

In $h_{(3,2)}\, e_{(2,2)}$, a monomial looks like

$$\left(x_{i_1^{(1)}}\, x_{i_2^{(1)}}\, x_{i_3^{(1)}}\right)\left(x_{i_1^{(2)}}\, x_{i_2^{(2)}}\right)\left(x_{j_1^{(1)}}\, x_{j_2^{(1)}}\right)\left(x_{j_1^{(2)}}\, x_{j_2^{(2)}}\right)$$

$$i_1^{(1)} \le i_2^{(1)} \le i_3^{(1)} \qquad i_1^{(2)} \le i_2^{(2)} \qquad j_1^{(1)} < j_2^{(1)} \qquad j_1^{(2)} < j_2^{(2)}$$

This corresponds to a biword

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & \bar{1} & \bar{1} & \bar{2} & \bar{2} \\ i_1^{(1)} & i_2^{(1)} & i_3^{(1)} & i_1^{(2)} & i_2^{(2)} & j_1^{(1)} & j_2^{(1)} & j_1^{(2)} & j_2^{(2)} \end{pmatrix}$$

No repitions!

Symmetric Function Identities

A biword like

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & \bar{1} & \bar{1} & \bar{2} & \bar{2} \\ 1 & 1 & 3 & 1 & 2 & 1 & 2 & 2 & 3 \end{pmatrix}$$

is taken by Super RSK to a pair

$$\left(\begin{array}{lllll} 3 & & & & \\ 2 & 2 & 3 & & \\ 1 & 1 & 1 & 1 & 2 \end{array} \; , \quad \begin{array}{lllll} \bar{2} & & & & \\ 2 & \bar{1} & \bar{2} & & \\ 1 & 1 & 1 & 2 & \bar{1} \end{array} \right)$$

SSYT SSMT'

(SSMT with entries of size one)

Symmetric Function Identities

Theorem

$$h_{\underline{a}} e_{\underline{b}} = \sum_{\lambda \vdash |\underline{a}|+|\underline{b}|} |SSMT'(\lambda, \underline{a}, \underline{b})| \, s_\lambda$$

SSMT with entries of size one.

Corollary

$$\mathrm{Sym}^{\underline{a}}(V_n) \otimes \Lambda^{\underline{b}}(V_n) \cong \bigoplus_{\lambda \vdash |\underline{a}|+|\underline{b}|} \left(W^{\lambda}_{GL_n}\right)^{\oplus |SSMT'(\lambda, \underline{a}, \underline{b})|}$$

Thank you!

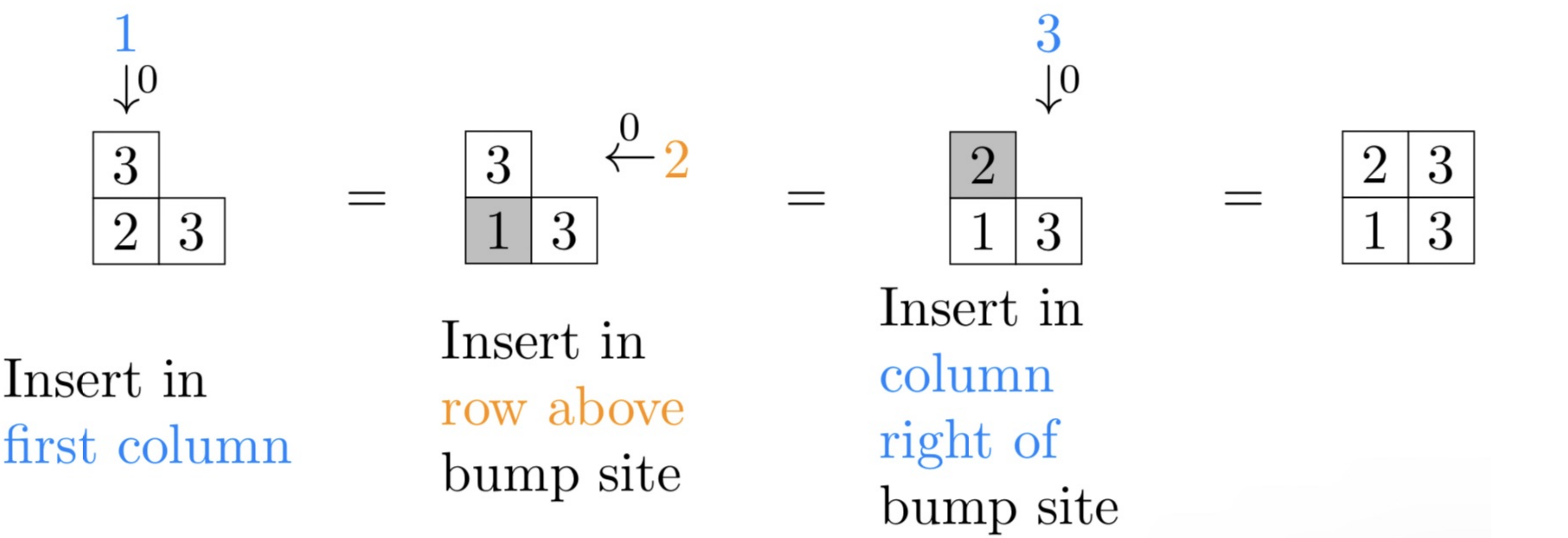
(wilsoa.github.io)

Super RSK

Super RSK (Muth 19) treats even and odd values seperately.

To perform 0-insertion, odd numbers are inserted in columns and even numbers are inserted in rows.

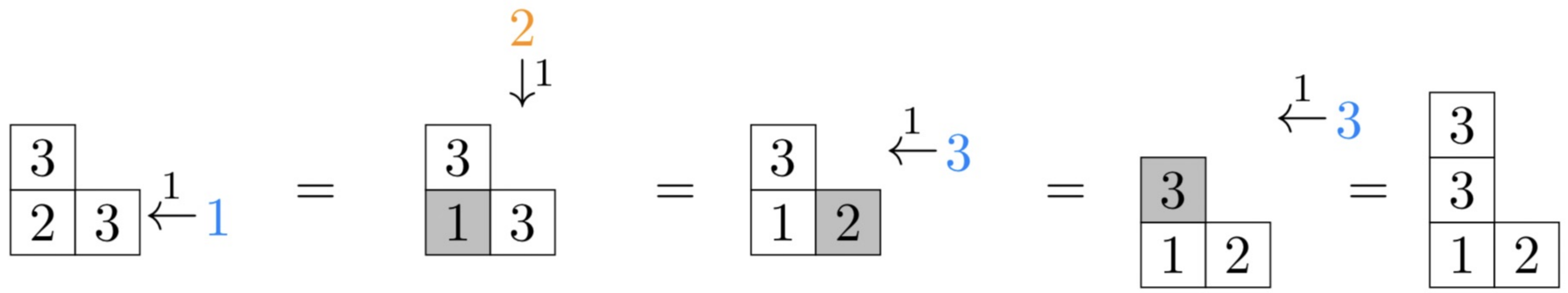
Ex)



Super RSK

To perform 1-insertion, even numbers are inserted in columns and odd numbers are inserted in rows

Ex)



Insert in first row

Insert in column right of bump site

Insert in row above bump site

Insert in row above bump site

To insert an array $\begin{pmatrix} a_1 & \cdots & a_\ell \\ b_1 & \cdots & b_\ell \end{pmatrix}$, 0-insert or 1-insert b_i if a_i is even or odd respectively.